Indian J. Phys. **43**, 320—323 (1969)

Note on the tosional vibration of a finite circular cylinder of non-homogeneous material by a particular type of twist on one of the plane surfaces.

By S. K. Biswas

10, *Broad Street, Calcutta*-19.

(*Received* 19 *May* 1969)

In this paper the torsional vibration of a cylinder of finite length of non-homogeneous material has been considered. The rigidity and density vary exponentially with depth. One end of the cylinder is fixed while a periodic shearing force acts along the circumference of a circle on the other end.

1. Introduction

Mitra & Sen Gupta (1967) solved the problem of torsional oscillation set up in a semi-infinite circular cylinder of non-homogeneous material by a particular type of twist on the plane end. In the present paper the author considers the torsional vibration set up in a finite circular cylinder of non-homogeneous material one of whose ends is fixed, the other having prescribed twist. The cylindrical coordinates (r, θ, z) are used with the origin on the free end and the z-axis which is drawn inside the cylinder coincides with the axis of the cylinder. The length of the cylinder is assumed to be l. The end $z=l$ is fixed and at the end $z=0$, a periodic hearinsg force acts along the circumference of a circle of radius b where $b<a$, a being the radius of the cylinder. It is assumed that the rigidity and the der.sityof the material of the cylinder vary exponentially with depth.

Problem, fundamental equation and boundary conditions.

Let us assume that $u_r = u_z = 0$ and u_θ $(= v)$ is independent of θ. The strain components are given by

$$e_{rr} = e_{\theta\theta} = e_{zz} = e_{zr} = 0,$$

$$e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \; e_{\theta z} = \frac{\partial v}{\partial z} \tag{1}$$

and the corresponding stresses are

$$\sigma_r = \sigma_\theta = \sigma_z = \sigma_{rz} = 0,$$

$$\sigma_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right),$$

$$\sigma_{\theta z} = \mu . \frac{\partial v}{\partial r}, \tag{2}$$

We suppose that the rigidity and the density of the material of the cylinder are given by

$$\mu = \mu_0 \, . \, e^{-\mu_1 z}$$
$$\rho = \rho_0 \, . \, e^{-\mu_1 z} \tag{3}$$

where ρ_0 , μ_0 , μ_1 are all constants, Evidently the velocity of the torsional waves ($=\sqrt{\mu/\rho}$) remains the same for all values of z .

Then

$$\sigma_{\theta z} = \mu_0 . \, e^{-\mu_1 z} \, . \, \frac{\partial v}{\partial z}$$
$$\sigma_{r\theta} = \mu_0 \, . \, e^{-\mu_1 z} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \tag{4}$$

Two equations of motion are satisfied identically and the remaining one reduces to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} - \mu_1 \frac{\partial v}{\partial z} = \frac{1}{c^2} \cdot \frac{\partial^2 v}{\partial t^2} \tag{5}$$

where $c = \sqrt{\mu_0/\rho_0}$. (6)

Boundary conditions are given by

i) $r\theta = 0$, when $r = a$;

ii) $v = 0$, when $z = l$;

iii) $(\theta z)_{z=0} = S . \, \delta \, (r-b) . \, e^{ipt}$, $0<b<a$ (7)

where S is a constant and δ is the Dirac's delta function.

3. Solution of the problem

If we take

$$v = R(r) . \, Z(z) . \, e^{ipt}, \tag{8}$$

equation (5) reduces to

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2}\right) + \frac{1}{Z}\left(\frac{d^2Z}{dz^2} - \mu_1 \frac{dZ}{dz}\right) = -p^2/c^2$$

Hence we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r} \cdot \frac{dR}{dr} - \frac{R}{r^2}\right) = -k^2, \tag{9}$$

$$\frac{1}{Z}\left(\frac{d^2Z}{dz^2} - \mu_1 \frac{dZ}{dz}\right) = k^2 - p^2/c^2 = \kappa^2, \text{ say}, \tag{10}$$

where we assume p to be very small so that $p^2/c^2 < k^2$.

The solution of (9) is, $R = \text{const.} \times J_1\ (kr)$, (11)

where J_1 is the first order Bessel function.

The solution of (10) is $Z = A.\ e^{qz} + B.\ e^{q'z}$ (12)

where $q = \dfrac{\mu_1 + \sqrt{\mu_1^2 + 4\alpha^2}}{2}$, $\quad q' = \dfrac{\mu_1 - \sqrt{\mu_1^2 + 4\alpha^2}}{2}$

and A, B are constants.

The first of the boundary conditions (7) gives $J_2\,(ka) = 0$. (13)

The roots of (13) are given by

$$k_1 a = 5{\cdot}136\,,$$
$$k_2 a = 8\,417\,,$$
$$k_3 a = 11{\cdot}6\,,$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

and $k_n a \approx (n + 3/4)\pi$, when n is large (Jahnke & Emde 1951).

The second of the boundary conditions (7) gives

$$\sum_{n=1}^{\infty} \{A_n.\ e^{q_n l} + B_n.\ e^{q'_n l}\}.\ J_1\,(k_n r).\ e^{ipt} = 0$$

Hence we may take

$$A_n.\ e^{q_n l} + B_n\ e^{q'_n l} = 0 \tag{14}$$

Now, we have

$$(\theta z)_{z=0} = \mu_0 \sum_{n=1}^{\infty} (q_n\ A_n + q'_n\ B_n).\ J_1\,(k_n r) e^{ipt}$$

The third of the boundary conditions (7) gives

$$\mu_0 \sum_{n=1}^{\alpha} (q_n\ A_n + q'_n\ B_n).\ J_1\,(k_n r) e^{ipt} = S.\ \delta\ (r-b).\ e^{ipt}$$

$$= f\,(r).\ e^{ipt}, \text{ say.} \tag{15}$$

Thus $f\,(r) = \sum_{n=1}^{\infty} \mu_0\ (q_n A_n + q_n' B_n).\ J_1(k_n r)$

Therefore $\int_0^a r\, f(r).\ J_1(k_n r)\, dr.$

$$= \mu_0 \left(q_n A_n + q'_n B_n\right) \int_0^a r \left[J_1(k_n r)\right]^2 dr$$

$$= \mu_0 \left(q_n A_n + q_n' B_n\right) \frac{a^2}{2} . \left[J_1(k_n a)\right]^2 \qquad \text{(Byerly 1959)}$$

$$\mu_0 \left(q_n A_n + q_n' B_n\right) = \frac{2}{a^2 \left[J_1(k_n a)\right]^2} \int_0^a r . f(r).\ (k_n r) dr. \tag{16}$$

From (15) we have

$$f\,(\text{r}) = S.\ \delta\,(r-b) \tag{17}$$

Using (17) in (16) and utilising the following property of Dirac delta function

$$\int_{-\infty}^{\infty} f\,(x).\ \delta\,(x-a) = f\,(a)$$

where $f(x)$ is continuous, we get

$$q_n A_n + q_n' B_n - \frac{2 S.b.\ J_1 (k_n b)}{\mu_0 .\ a^2 [\, J_1 (k_n a)]^2} = 0 \tag{18}$$

From (14) and (18), we get

$$A_n = \frac{-2. S.b.\ J_1 (k_n b).\ e^{q'_n l}}{\mu_0 .\ a^2 .\ [J_1(k_n a)]^2\, [q'_n e^{q'_n l} - q_n e^{q'_n l}} \tag{19}$$

$$B_n = \frac{2. S.b.\ J_1 (k_n b).\ e^{q_n l}}{\mu_0 .\ a^2 .\ [J_1(k_n a)]^2\, [q'_n e^{q_n l} - q_n e^{q'_n l}} \tag{20}$$

Hence from (8), (11), (12), (19), and (20) we get finally

$$v = \frac{2 S.l}{\mu_0 .\ a^2} \sum_{n=1}^{\infty} \frac{J_1(k_n b)[e^{q_n l}. e^{q'_n z} - e^{q'_n l}. e^{q_n z}]}{[J_1(k_n a)]^2 [q'_n e^{q_n l} - q_n e^{q_n l}]} \times J_1(k_n r). e^{ipt}. \tag{21}$$

The series in (21) can be verified to be a convergent series.

The author takes this opportunity to express his sincere thanks to Dr. A. K. Mitra, Reader in Mathematics, Jadavpur University for his help and guidance in the preparation of this paper.

References

Byerly, W. E. 1959 *An elementary treatise on Fourier series and spherical, cylindrical and ellipsoidal Harmonics*, Dover Publication.

Jahnke, E. & Emde, F. 1951 *Tables of functions with formulas and curves*, 4th Edition, Dover Publication

Mitra, A. K. & Sen Gupta, A. K. 1967 *Indian Jour, of Mechanics and Math*, **5**, 43.